

ANALYTIC APPROACH TO S^1 -EQUIVARIANT MORSE INEQUALITIES

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ABSTRACT. It is well known that the cohomology groups of a closed manifold M can be reconstructed using the gradient dynamical of a Morse-Smale function $f: M \rightarrow \mathbb{R}$. A direct result of this construction are Morse inequalities that provide lower bounds for the number of critical points of f in term of Betti numbers of M . These inequalities can be deduced through a purely analytic method by studying the asymptotic behaviour of the deformed Laplacian operator. This method was introduced by E. Witten and has inspired a numbers of great achievements in Geometry and Topology in few past decades. In this paper, adopting the Witten approach, we provide an analytic proof for; the so called; equivariant Morse inequalities when the underlying manifold is acted on by the Lie group $G = S^1$ and the Morse function f is invariant with respect to this action.

1. INTRODUCTION

Classical Morse theory with all its variants are amongst the great achievement of the modern geometry and topology. Its relevance to topology begun by the fundamental observation that the cellular structure of a closed manifold M can be reconstructed through level sets of a Morse function f on M . In particular this give a way to reconstruct the cellular chain complex and therefore the cohomology of M , as is explained clearly in [8] and [14].

The seminal paper of E. Witten [16] which was inspired by ideas from quantum field theory, shed a new light on Morse theory by providing a new chain complex for computing the cohomology of M . This complex; called Morse-Smale-Witten complex; is generated by the critical points of f and graded by their Morse indices, as in the cellular complex. However its differentials are defined through the gradient lines between critical points whose indices differ by one. We refer to [14] for a detailed exposition of this theory. Amongst others, this construction led to the innovation of Floer Homology and solved (partially) the Arnold conjecture, c.f. [13].

An immediate consequence of the reconstruction of the singular cohomology via critical points of a Morse function is the Morse inequalities. Roughly speaking, they provide lower bounds for the number of critical points in term of the Betti numbers of M , i.e. the rank of the cohomology spaces of M . As far as one is interested in these inequalities rather than the cohomology itself, there is a very elegant and conceptual analytic derivation. This is the Witten idea of deforming the de Rham complex in an appropriate way using the Morse function and then study the asymptotic behaviour of this complex. In this paper we follow Roe's account of Witten's approach in [11, chapter 14].

Morse theory can be generalized in other directions. For instance in some situation there is a Lie group G acting on M and preserves f . This problem naturally arises in n -body problem where the central configurations are critical points of some Morse function which are symmetric with respect to the action of $SO(n)$ (c.f. [10]). Another example is the problem of finding the number of closed geodesics of a Riemannian metric where the Lie group is S^1 . Actually this last example was amongst the first applications of Morse theory that was worked out by Morse in [9], see also [8, chapter 3]. In these cases the critical levels of f are clearly orbits of the action. The Morse-Bott theory ignore the invariance of f under the action and provides lower-bounds for the number of these critical levels (see [5] and [6, page 344]). However; as it is clearly explained in [6, pages 351-355]; to get best results one has to assume the G -invariance of f and this requires an appropriate cohomology theory that takes the group action into its construction. This is the *equivariant cohomology theory* which is introduced originally by E. Cartan in 1940s. In this paper we will deal with A. Borel definition of equivariant de Rham complex as is explained; amongst other applications of equivariant cohomology; in [2] and [7]. As in ordinary Morse-Bott theory, the equivariant cohomology may be reconstructed from a cellular

chain complex generated by critical level of f . This is done by A. G. Wasserman in [15, section 4] for a compact Lie group G . Nevertheless the Morse-Smale-Witten complex for equivariant cohomology has been worked out recently, for $G = S^1$, by M. J. Berghoff in his PhD thesis [4]. These constructions lead to the equivariant Morse inequalities whose precise statement might be found in [15, page 149] or in [6, page 351].

In this paper we consider also the case $G = S^1$ and prove the equivariant Morse inequalities by adopting the Witten method to deform the Borel complex. This approach will be interesting for those who are interested in analytical methods rather than in topological ones. Moreover this analytic approach is more flexible and can be adopted to different situations. One of the authors has already used this approach to deal with Morse inequalities on manifolds with boundary and, the so called, delocalized Morse inequalities, c.f. [17] and [18].

The structure of the paper is as follows. In section 2 we give the technical definition of the equivariant cohomology theory by introducing the Borel complex. Then we give the precise statement of the equivariant Morse inequalities by stating the main theorem 2.1. In section 3 we establish the Hodge theory for the equivariant complex and introduce the Witten deformation of the Borel complex. Then we prove an infinite number of quite general equivariant analytical Morse inequalities in theorem 3.2. The asymptotic behaviour of these inequalities leads finally to the Morse inequalities. The lemma 3.1 will be used in last section to show that the number of Morse inequalities is actually finite. The key lemma 3.5 reduces our problem to compute the kernel of some elliptic operators on Euclidian spaces. The computation of these kernels are the subject of section 4 and lead to the proof of the main theorem in last paragraph of the paper.

2. EQUIVARIANT COHOMOLOGY AND MAIN THEOREM

Let $G = S^1$ be a compact Lie group that acts on a topological space X and let EG be a contractible space that is acted on, freely and continuously, by G . Such spaces exist and are unique up to homotopy equivalence and the quotient $BG := EG/G$ is the classifying space of G . The diagonal action of G on $X \times EG$ is free and the quotient $X_G := (X \times EG)/G$ is called the homotopy path space. The equivariant cohomology of X ; that we denote by $H_G^*(X)$; is by definition the singular cohomology (with coefficient in C) of the homotopy path space

$$(2.1) \quad H_G^*(X) = H^*(X_G)$$

If G acts trivially then $X_G = X \times BG$ and $H_G^*(X) = H^*(X \times BG) = H^*(X) \otimes H^*(BG)$. In particular the Equivariant cohomology of a single point is the cohomology of the group, i.e. $H^*(G; \mathbb{R})$. If the action of G on X is free then $EG \rightarrow X_G \rightarrow X/G$ is a fibration with contractible fibres which implies $H_G^*(X) = H^*(X/G)$. Actually the equivariant cohomology gives a way to combine the cohomology of the space X and that of G while taking into the account the action of G on X . When X is a differential manifold (that we denote by M from now on) and the action is smooth there is a more geometric approach to the construction of the equivariant cohomology. Because our study concerns this formulation, we give a short description of this construction. We consider the simplest case when the Lie group G is just the circle S^1 .

Let M be a closed smooth manifold of dimension n which is acted on by the group $G = S^1$. This action is supposed to be smooth and not necessarily free. The Lie algebra of S^1 is \mathbb{R} with a fixed element 1. This element generates a vector field v over M which is tangent to the orbits and vanishes at fixed points of the action. The vector field v is called the infinitesimal generator of the action and we denote its t -time flow by ϕ_t . Let $\Omega_G^*(M) \subset \Omega^*(M)$ consists of all invariant differential forms ω satisfying $\phi_t^*(\omega) = \omega$ for $t \in \mathbb{R}$, or equivalently $\mathcal{L}_v(\omega) = 0$. Consider the algebra $\mathbb{C}[t] \otimes \Omega_G^*(M)$. This is indeed the algebra of all polynomial function on the Lie algebra $\mathfrak{g} = \mathbb{R}$ with values in $\Omega_G^*(M)$. Here $t \in \mathfrak{g}^*$ is dual element corresponding to 1, i.e. $t(1) = 1$. This algebra is graded by the rule $\deg(t^k \otimes \omega) = 2k + \deg(\omega)$. Put $\Omega_{eq}^*(M) := \mathbb{C}[t] \otimes \Omega_G^*(M)$ then the following linear map

$$(2.2) \quad \begin{aligned} d_{eq} : \Omega_{eq}^*(M) &\rightarrow \Omega_{eq}^*(M) \\ d_{eq}(t^k \otimes \omega) &= t^k \otimes d\omega + t^{k+1} \otimes i_v \omega \end{aligned}$$

is a differential, i.e. $d_{eq}^2 = 0$ and increases the degree by one. The *equivariant de Rham cohomology groups* $H_G^*(M)$ are the cohomology groups of this graded differential complex. It turns out that these groups are isomorphic to the groups introduced by (2.1) when X is a smooth manifold.

The cohomology space $H_G^k(M)$ is a complex finite dimensional vector space. So one can define the equivariant Betti numbers by $\beta_G^k := \dim H_G^k(M)$. When the action $S^1 \times M \rightarrow M$ has no fixed points, then $\tilde{M} = M/S^1$ is a smooth manifold and equivariant cohomology groups of M are canonically isomorphic to the de-Rham cohomology of \tilde{M} . The equivariant cohomology of a point is just the algebra $\mathbb{R}[t]$ (with $\deg t = 2$) which is isomorphic to the de-Rham cohomology of $BS^1 = \mathbb{C}P^\infty$ when t is identified to a symplectic form on $\mathbb{C}P^\infty$.

Let $f : M \rightarrow \mathbb{R}$ be an invariant smooth function, i.e. $v.f = 0$. An orbit o is critical if one point on it (then all points) is critical point for f . For $x \in o$ let N_x stands for the quotient space $T_x M / T_x o$. For $x \in M$ and $X, Y \in T_x M$ the Hessian of f is a symmetric bi-linear form defined as follows

$$H_f(X, Y) = X.(Y.f) - (\nabla_X Y).f$$

Here ∇ is the Riemannian connection on TM associated to a Riemannian metric g . Because S^1 is compact it is always possible, through an averaging procedure, to assume g be S^1 -invariant. With this assumption it is true that if X or Y belong to $T_x o$ then $H_f(X, Y) = 0$. Therefore the Hessian defines a well defined symmetric bi-linear form on N . Using the Riemannian metric, we can identify N_x with the orthogonal complement of $T_x o$. We denote the restriction of H to $N \subset TM$ by \bar{H}_f . We say a critical orbit o be transversally non-degenerated (or simply non-degenerated) if \bar{H}_f is non-degenerated at any points of o . The Morse index of such orbit is the dimension of the maximal subspace of N_x , on which the Hessian is negative-definite. In the sequel we reserve the notation o for a non-trivial orbit and we denote a trivial orbit by its geometric image, that is a point p in M . Let c_k and d_k denote respectively the number of critical points and orbits with Morse index k . Our aim is to provide an analytic proof for the following equivariant Morse inequalities via Witten deformation:

Theorem 2.1. *With $\tilde{c}_k := d_k + c_k + c_{k-2} + c_{k-4} + \dots$ the following inequalities hold for $k = 0, 1, 2, \dots$*

$$\tilde{c}_k - \tilde{c}_{k-1} + \dots \pm \tilde{c}_0 \geq \beta_{eq}^k - \beta_{eq}^{k-1} + \dots \pm \beta_{eq}^0,$$

Actually the inequalities for $k \geq n+1$ are the same that the inequality for $k = n$.

Note that, when the action is free these inequalities reduce to the ordinary Morse inequalities for the function $\tilde{f} : \tilde{M} \rightarrow \mathbb{R}$, while it reduces to the Morse inequalities for the function f when the action is trivial.

3. EQUIVARIANT HODGE THEORY AND ANALYTIC MORSE INEQUALITIES

For our purposes in this paper, we need to establish an equivariant version of Hodge theory. The space of S^1 -invariant differential forms $\Omega_G^*(M)$ is endowed with an inner product coming from the Riemannian metric g and its natural lifting to the exterior algebras $\wedge^*(T_p M)$ for all $p \in M$. The formal dual of the exterior differential d is the operator d^* . We define a scalar product on $\mathbb{C}[t] \otimes \Omega_G^*(M)$ by the bi-linear extension of the following formula

$$(3.1) \quad \langle t^i \otimes \omega, t^j \otimes \eta \rangle = \delta_{ij} \langle \omega, \eta \rangle$$

It is easy to verify that the formal adjoint of the equivariant exterior derivative d_{eq} (see (2.2)) with respect to this scalar product is the linear extension of the following operator of grade -1 , where $\epsilon_{0i} = 1 - \delta_{0i}$ throughout this paper

$$(3.2) \quad \begin{aligned} d_{eq}^* : \Omega_{eq}^*(M) &\rightarrow \Omega_{eq}^*(M) \\ d_{eq}^*(t^i \otimes \omega) &= t^i \otimes d^*(\omega) + \epsilon_{0i} t^{i-1} \otimes v^* \wedge \omega \end{aligned}$$

The equivariant Laplacian operator is defined by $\Delta_{eq} := d_{eq}^* d_{eq} + d_{eq} d_{eq}^*$. The following relation is a very direct consequence of definitions and give the action of Δ_{eq} on a term like $t^i \otimes \omega$. Here Δ stands for the laplacian operator on differential forms.

$$(3.3) \quad \begin{aligned} \Delta_{eq}(t^i \otimes \omega) &= t^i \otimes (\Delta \omega + v^* \wedge i_v \omega + \epsilon_{0,i} i_v v^* \wedge \omega) \\ &\quad + t^{i+1} \otimes (i_v d^* \omega + d^* i_v \omega) \\ &\quad + \epsilon_{0i} t^{i-1} \otimes dv^* \wedge \omega \end{aligned}$$

Clearly Δ_{eq}^k may be considered as a differential operator on $\Omega_{eq}^k(M)$ consisting of all smooth sections of the equivariant exterior algebra $\wedge_{eq}^* T_p M$ where

$$\wedge_{eq}^k T_p M := \bigoplus_{i+j=k} t^i \otimes \wedge^j T_p$$

From this point of view, it is clear that Δ_{eq}^k is a second order elliptic differential operator which is formally self adjoint with respect to above scalar product. Note that here t is merely a label to keep track of the grading and $\wedge_{eq}^k TM$ is a finite dimensional hermitian vector bundle. So, we may apply the ordinary completion procedure to construct the Hilbert spaces $L^2(M, \wedge_{eq}^k TM)$ or Sobolev spaces $W^\alpha(M, \wedge_{eq}^k TM)$. Therefore, exactly as in the classical Hodge theory for the Laplacian (e.g. through the construction of heat operator as in [12, chapter 3]) we have the following isomorphisms for $k = 0, 1, 2, \dots$ (see [12, Theorem 1.45])

$$(3.4) \quad H_G^k(M) \simeq \ker \Delta_{eq}^k$$

It is clear from expansion (3.3) that for $k \geq n-1$, multiplication by t^i gives rise to the isomorphism $\Omega_{eq}^k(M) \simeq \Omega_{eq}^{k+2i}(M)$ and the action of the Laplacian is linear with respect to this multiplication. Therefore $t^i \otimes \ker \Delta_{eq}^k = \ker \Delta_{eq}^{k+2i}$ which implies, by (3.4), the following results for $k \geq n-1$

$$(3.5) \quad H_G^k(M) \simeq H_G^{k+2i}(M)$$

We use the following lemma to prove that the equivariant Morse inequalities stop beyond order n .

Lemma 3.1. *For $k \geq n$ the following equalities hold*

$$\beta_{eq}^k - \beta_{eq}^{k+1} = (-1)^k \chi(M)$$

where $\chi(M)$ is the Euler characteristic of M .

Proof Using (3.5) it is enough to prove the lemma for $k = n$. It is clear from (2.2) that d_{eq} is $\mathbb{C}[t]$ -linear. However; due to the term ϵ_{0i} in (3.2); the operator d_{eq}^* is $\mathbb{C}[t]$ -linear on Ω_{eq}^k only for $k \geq n$. Because $t \otimes \Omega_{eq}^{n-1}(M) = \Omega_{eq}^{n+1}$ and $t \otimes \Omega_{eq}^n(M) = \Omega_{eq}^{n+2}$ we can define the following operator

$$\begin{aligned} \bar{D}_{eq} : \Omega_{eq}^n \oplus \Omega_{eq}^{n+1} &\rightarrow \Omega_{eq}^n \oplus \Omega_{eq}^{n+1} \\ \bar{D}_{eq} &= \bar{d}_{eq} + \bar{d}_{eq}^* \end{aligned}$$

Here $\bar{d}_{eq} := d_{eq}$ and $\bar{d}_{eq}^* := t \otimes d_{eq}^*$ on the first summand while $\bar{d}_{eq} := t^{-1} \otimes d_{eq}$ and $\bar{d}_{eq}^* := d_{eq}^*$ on the second summand. The operators \bar{d}_{eq} and \bar{d}_{eq}^* are formal adjoint of each other with respect to the inner product (3.1). Therefore the *equivariant de Rham operator* \bar{D}_{eq} is grading reversing and formally self adjoint. It is clear from the definitions that $\bar{D}_{eq}^2 = \Delta_{eq}$ therefore \bar{D}_{eq} is an elliptic differential operator and its Fredholm index is given by

$$\text{ind } \bar{D}_{eq} = \dim \ker \Delta_{eq}^n - \dim \ker \Delta_{eq}^{n+1} = \beta_{eq}^n - \beta_{eq}^{n+1}$$

On the other hand, as differential operators on $\Omega_{eq}^n \oplus \Omega_{eq}^{n+1}$, the elliptic operators \bar{D}_{eq} and the classical de Rham operator $D := d + d^*$ have the same principal symbols. Therefore they are homotopic in the space of Fredholm operators. Because Fredholm index is homotopy invariant, we get the following equality

$$\text{ind } \bar{D}_{eq} = \text{ind } D = \beta^n + \beta^{n-2} + \dots - \beta^{n-1} - \beta^{n-3} - \dots$$

where β^i is the i -th Betti number of M . In above the right hand side equals $(-1)^n \chi(M)$ by definition and this completes the proof of the lemma. \square

What we need is a deformed version of the above equivariant Laplacian that we introduce here. Given a positive parameter s and using the Morse function f , the Witten deformation of d_{eq} is the operator $d_{eq,s} := e^{-sf} d_{eq} e^{sf} = d_{eq} + s df \wedge$ and its formal dual is $d_{eq,s}^* = d_{eq}^* + s df \lrcorner$. The associated deformed equivariant Laplacian $\Delta_{eq,s} := d_{eq,s}^* d_{eq,s} + d_{eq,s} d_{eq,s}^*$ has the following expansion, c.f. [11, Lemma 9.17]

$$(3.6) \quad \Delta_{eq,s} = \Delta_{eq} + s^2 |df|^2 + s H_f$$

Here H_f is the following operator, where $\{e_i\}_i$ is a local orthonormal base for TM and L_{e_i} and R_{e_i} are respectively the left and right Clifford multiplication by e_i (see [11, page 124])

$$(3.7) \quad H_f = \sum_{i,j} H_f(e_i, e_j) L_{e_i} R_{e_j}$$

It is clear that $\Omega_{eq}^*(M)$ with the differential $d_{eq,s}$ is a graded differential complex, so we may define the deformed cohomology spaces $H_{eq,s}^k(M)$. Nevertheless these cohomology spaces are not new object and multiplication by e^{-sf} provides the following isomorphisms

$$(3.8) \quad H_{eq}^k(M) \simeq H_{eq,s}^k(M).$$

Using the expansion (3.6) and Duhamel's formula, one is able to construct the heat operator associated to $\Delta_{eq,s}$ which establishes a Hodge theory and provides the isomorphisms $\ker \Delta_{eq,s}^k \simeq H_{eq,s}^k(M)$. This and (3.8) give the following relation

$$(3.9) \quad \beta_{eq}^k = \dim(\ker \Delta_{eq,s}^k)$$

By (3.6), $\Delta_{eq,s}^k$ is a positive and second order elliptic differential operator on $L^2(M, \wedge_{eq}^k TM)$. Therefore given a smooth rapidly decreasing positive function ϕ on $\mathbb{R}^{\geq 0}$, the operator $\phi(\Delta_{eq,s}^k)$, being a smoothing operator on $L^2(M, \wedge_{eq}^k TM)$ is a trace class operator. Its trace is denoted by

$$\mu_{eq,s}^k = \text{tr } \phi(\Delta_{eq,s}^k); \quad k = 0, 1, \dots, n$$

The following equivariant analytic Morse inequalities are our departure point for a proof of theorem 2.1 (see [11] for the non-equivariant version)

Theorem 3.2 (the analytic equivariant Morse inequalities). *With the above notations, the following inequalities hold*

$$\mu_{eq,s}^k - \mu_{eq,s}^{k-1} + \dots \pm \mu_{eq,s}^0 \geq \beta_{eq}^k - \beta_{eq}^{k-1} + \dots \pm \beta_{eq}^0$$

Proof If we put $\beta_{eq,s}^k = \dim(\ker \Delta_{eq,s}^k)$ then by (3.9) the above inequalities are equivalent to the followings

$$\mu_{eq,s}^k - \mu_{eq,s}^{k-1} + \dots \pm \mu_{eq,s}^0 \geq \beta_{eq,s}^k - \beta_{eq,s}^{k-1} + \dots \pm \beta_{eq,s}^0$$

The proof of the proposition 14.3 of [11] can be applied literary to the deformed Laplacian and gives these inequalities. For the seek of completeness we give a very brief account of this proof. The

spectrum of the deformed Laplacian $\Delta_{eq,s}^k$ is discrete, so there is a rapidly decreasing function $\tilde{\phi}$ on \mathbb{R} which vanishes on non-zero elements of the spectrum such that $\tilde{\psi}(0) = 1$. Therefore $\beta_{eq,s}^k = \text{tr } \tilde{\phi}(\Delta_{eq,s}^k)$ which implies $\mu_{eq,s}^k - \beta_{eq,s}^k = \text{tr}(\phi - \tilde{\phi})(\Delta_{eq,s}^k)$. The relation $(\phi - \tilde{\phi})(x) = x\psi(x)^2$ defines a rapidly decreasing function ψ on \mathbb{R} and one get the following

$$\mu_{eq,s}^k - \beta_{eq,s}^k = \text{tr } \Delta_{eq,s}^k \psi(\Delta_{eq,s}^k)^2$$

Let H_j denotes the L^2 -Hilbert space generated by $\Omega_{eq}^j(M)$. Using the relation $\Delta_{eq,s}^j = d_{eq,s}^* d_{eq,s} + d_{eq,s} d_{eq,s}^*$ one gets easily the following relation

$$\text{tr}\{d_{eq,s} d_{eq,s}^* \psi(\Delta_{eq,s}^j)^2\}_{|H_j} = \text{tr}\{d_{eq,s}^* d_{eq,s} \psi(\Delta_{eq,s}^{j-1})^2\}_{|H_{j-1}}$$

An alternating summation from $j = k$ to $j = 0$ on this relation implies the following relation

$$\mu_{eq,s}^k - \mu_{eq,s}^{k-1} + \cdots \pm \mu_{eq,s}^0 = \text{tr}\{d_{eq,s}^* d_{eq,s} \psi(\Delta_{eq,s}^k)^2\}_{|H_k}$$

Since $d_{eq,s}^* d_{eq,s} \psi(\Delta_{eq,s}^k)^2$ is a non-negative operator, the right side of the above relation is non-negative in general and this gives the equivariant Morse inequalities. \square

By the above theorem, to prove the theorem 2.1 we need just to study the asymptotic behaviour of $\mu_{eq,s}^k$ when s goes toward infinity. Since ϕ is rapidly decreasing, the operator $\phi(\Delta_{eq,s}^k)$ is smoothing and has a smooth kernel

$$\phi(\Delta_{eq,s}^k)(\omega)(p) = \int_M K_s^k(p, q) \omega(q) d\mu_g(q)$$

Here $K_s^k(p, q)$ is an element of $\wedge_{eq}^k T_p M \otimes \wedge_{eq}^k T_q^* M$. Therefore we have

$$(3.10) \quad \mu_{eq,s}^k = \int_M \text{tr } K_s^k(p, p) d\mu_g(p)$$

In a complement of an open neighbourhood of critical level (i.e. a critical point or orbit) of f we have $|df| \geq c > 0$ and the term $s^2 |df|^2$ in (3.6) dominates the other terms when s goes to infinity. Therefore, informally saying, on sections supported in this set the spectrum of $\Delta_{eq,s}^k$ goes to infinity. This makes the following lemma true. The proof of this lemma for the non-equivariant case can be found in [11, Lemma 14.6]. This proof uses the finite propagation speed property of the wave operator and Friedrich extension theorem. It can be adopted without problem to our context.

Lemma 3.3. *When s goes toward infinity, the smoothing kernel $K_s^k(p, q)$ goes uniformly to zero when p or q belong to a complement of an open neighbourhood of the critical levels of f . Consequently, by (3.10), to study the asymptotic behavior of $\mu_{eq,s}^k$ when s goes toward infinity, we need just consider the contribution of points p in any arbitrary small open neighbourhood of critical levels of f .*

For $\rho > 0$ let $N_{4\rho}(p)$ and $N_{4\rho}(o)$ denote, respectively, the 4ρ -neighbourhood of the critical point p and the critical orbit o . Let also ϕ_p and ϕ_o denote equivariant non-negative smooth functions on M which are supported, respectively, in $N_{3\rho}(p)$ and in $N_{3\rho}(o)$ such that $\phi_p = 1$ on $N_{\rho}(p)$ and $\phi_o = 1$ on $N_{\rho}(o)$. Point-wise multiplication by these functions defines operators on equivariant differential forms. The following corollary comes up as a very direct result of the above lemma:

Corollary 3.4. *The following relation holds*

$$\lim_{s \rightarrow \infty} \text{tr } \phi(\Delta_s^k) = \sum_p \lim_{s \rightarrow \infty} \text{tr}(\phi_p \phi(\Delta_s^k)) + \sum_o \lim_{s \rightarrow \infty} \text{tr}(\phi_o \phi(\Delta_s^k))$$

Let $\mathbb{B}_a^n(0)$ denote the ball in \mathbb{R}^n with center 0 and radius a . It is clear that $N_{4\rho}(p)$ and $\mathbb{B}_{4\rho}^n(0)$ are naturally isometric and in this isometry the point p corresponds to 0. This is also true for $N_{4\rho}(o)$ and $S^1 \times \mathbb{B}_{4\rho}^{n-1}(0)$ where o corresponds to $S^1 \times \{0\}$. Let L_s^k and \mathcal{L}_s^k denote, respectively, differential operators on $\Omega_{eq}^k(\mathbb{R}^n)$ and $\Omega_{eq}^k(S^1 \times \mathbb{R}^{n-1})$ such that with respect to above isometries

$$\Delta_{s|N_{4\rho}(p)}^k = L_{s|\mathbb{B}_{4\rho}^n(0)}^k \quad \text{and} \quad \Delta_{s|N_{4\rho}(o)}^k = \mathcal{L}_{s|S^1 \times \mathbb{B}_{4\rho}^{n-1}(0)}^k$$

Then; through a standard argument; based on Fourier inversion formula and finite propagation speed of wave operators; the following equalities hold

$$\phi(\Delta_s^k)(\omega_1) = \phi(L_s^k)(\omega_1) \quad \text{and} \quad \phi(\Delta_s^k)(\omega_2) = \phi(\mathcal{L}_s^k)(\omega_2)$$

provided that the Fourier transform $\hat{\phi}$ of ϕ is supported in $(-\rho, \rho)$ and the support of ω_1 and ω_2 are included, respectively, in $\mathbb{B}_{3\rho}^n(0)$ and $\mathbb{S}^1 \times B_{3\rho}^{n-1}(0)$. Therefore

$$\text{tr}(\phi_p \phi(\Delta_s^k)) = \text{tr}(\phi_p \phi(L_s^k)) \quad \text{and} \quad \text{tr}(\phi_o \phi(\Delta_s^k)) = \text{tr}(\phi_o \phi(\mathcal{L}_s^k))$$

These equalities and corollary 3.4 together prove the following lemma

Lemma 3.5. *Provided that the support of $\hat{\phi}$; the Fourier transform of ϕ ; is included in a sufficiently small neighbourhood of $0 \in \mathbb{R}$ and with above notations, the following relation holds*

$$\lim_{s \rightarrow \infty} \mu_s^k = \lim_{s \rightarrow \infty} \text{tr} \phi(\Delta_s^k) = \sum_p \lim_{s \rightarrow \infty} \text{tr}(\phi_p \phi(L_s^k)) + \sum_o \lim_{s \rightarrow \infty} \text{tr}(\phi_o \phi(\mathcal{L}_s^k))$$

Here p runs over all critical points of f while o runs over all critical orbits of f and L_s^k and \mathcal{L}_s^k are the local representation of Δ_s^k around 4ρ -neighbourhood of, respectively, critical points and critical orbits.

4. LOCALIZATION ON CRITICAL LEVELS

To compute the contribution of critical levels, we need to have a good representation of the deformed equivariant Laplacian operators around them. This is provided by an equivariant version of the Morse lemma that we are going to explain. Let us begin with an equivariant version of the tubular neighbourhood theorem. Suppose that G be a compact Lie group acting on the closed manifold M and g be a Riemannian metric which is invariant under the action. The map $f_x : G/G_x \rightarrow G.x$ given by $f_x([h]) = h.x$ is a diffeomorphism, where $G.x$ and G_x are the orbit and stablizer of $x \in M$. For $h \in G_x$, the derivative $T_x h : T_x M \rightarrow T_x M$ is an isometry that maps $T_x G.x$ into itself. Therefore, it induces a linear isometry $\phi(h) : N_x \rightarrow N_x$ where $N_x \subset T_x M$ is the orthogonal complement of $T_x G.x$. In other words one has an orthonormal representation $\phi : G_x \rightarrow O(N_x)$. The subgroup G_x has a free action on $G \times N_x$ given by $h.(h', v) = (h'h^{-1}, \phi(h)(v))$. The quotient space is a vector bundle $\pi : N \rightarrow G.x$ whose fibers are isometric to N_x (here we have used the identification $G_x = G/G_x$). The action of G on $G \times N_x$; given by $h.(h', v) = (hh', v)$; commutes with the action of G_x . Therefore it induces a bundle map on the vector bundle $N \rightarrow G.x$. The equivariant tubular neighbourhood theorem [3, page 21] asserts that there is an invariant neighbourhood W of G/G_x ; as the zero section of the bundle N ; and an invariant neighbourhood U of the orbit $G.x$ and an equivariant diffeomorphism $\bar{f} : W \rightarrow U$ that extends the orbit map f and makes the following diagram commutative

$$\begin{array}{ccc} G/G_x & \xrightarrow{f} & G.x \\ i \downarrow & & i \downarrow \\ W \subset N & \xrightarrow{\bar{f}} & U \subset M \end{array}$$

A particular case of this theorem is when $G_x = G$, i.e. the orbit is a point x . In this case the above theorem provides a coordinates system around x with respect to which the elements of G act as orthogonal maps. When $G = S^1$ and $G.x = S^1$ (i.e. $G_x = \mathbb{Z}_m$) then $N = S^1 \times \mathbb{R}^{n-1}$ and the action of S^1 factors as $e^{i\theta}.(e^{i\psi}, v) = (e^{i(m\theta+\psi)}, \phi(e^{i\theta})v)$ where $\phi : S^1 \rightarrow SO(n-1)$ is a homomorphism. Although the following lemma should exists in the literatures, we did not find it, so we provide a proof for it.

Lemma 4.1 (Equivariant Morse Lemma). *Let x be a critical point of the invariant Morse function f and also a fixed point of the action. There is a coordinate system (x_1, x_2, \dots, x_n) around x that maps x to 0 such that*

$$f(x_1, x_2, \dots, x_n) = \pm x_1^2 \pm x_2^2 \pm x_{k+1}^2 \pm \dots \pm x_n^2$$

and the action of G in this coordinates goes through orthogonal maps on \mathbb{R}^n . The case of a critical orbit reduces to this case just by restricting to the normal direction.

Proof We use the Moser approach to this problem as it is presented in [1, page 176]. Without loose of generality we may assume $G \subset O(n)$ and 0 is the critical point of f such that $f(0) = 0$. Consider the following differential one-forms on \mathbb{R}^n

$$\omega_0(x) = df(x) ; \quad \omega_1(x) = D^2f(0)(x, \cdot)$$

Clearly $\omega_1 = dh$ where $h(x) = \frac{\partial^2 f}{2\partial x_i \partial x_j}(0)x_i x_j$. Let z_t be a time-depending smooth vector field around 0 and let ϕ_t stands for its flow. The following relations hold where $\omega_t = (1-t)\omega_0 + t\omega_1$

$$\begin{aligned} \frac{d}{dt}\phi_t^*\omega_t &= \phi_t^*(\mathcal{L}_{z_t}\omega_t + \frac{d\omega_t}{dt}) \\ &= \phi_t^*d(i_{z_t}\omega_t + (h-f)). \end{aligned}$$

Since $df(0) = 0$ and $D^2f(0)$ is non-singular, the one-forms ω_t are non-degenerate in a neighbourhood of 0. Therefore the algebraic equation $i_{z_t}\omega_t + (h-f) = 0$ has unique solution for vector field z_t with $z_t(0) = 0$. Thus for this vector field $\phi_1^*\omega_1 = \phi_0^*\omega_0 = \omega_0$ and $f \circ \phi_1^{-1}(x) = h(x) = \frac{1}{2}\partial_i\partial_j f(0)x_i x_j$. The new point here is that f and h are invariant and since this action is linear ω_t is also invariant. Therefore, z_t should be invariant and its flow Φ_t should be equivariant. This gives the equivariant Morse lemma after applying appropriate orthogonal transformation which diagonalize h . \square

For next uses we need to give explicit expressions for the action and Morse function around both critical points and orbits. Let p be a fixed point of the action of S^1 on M and the critical point of Morse function f . Then there is a coordinate system $x = (x_1, \dots, x_{2q-1}, x_{2q}, \dots, x_n)$ centred at p that satisfies the following conditions.

- In this coordinate system the metric g is given by $g = dx_1^2 + dx_2^2 + \dots + dx_n^2$.
- The action of S^1 is as follows

$$(4.1) \quad e^{i\theta}(x_1 + ix_2, \dots, x_n) = (e^{im_1\theta}(x_1 + ix_2), \dots, e^{im_q\theta}(x_{2q-1} + ix_{2q}), x_{2q+1}, \dots, x_n)$$

where $m_i \in \mathbb{N}$.

- The Morse function takes the following form

$$(4.2) \quad f(x_1, x_2, \dots, x_{2q+1}, \dots, x_n) = \epsilon_1(x_1^2 + x_2^2) + \dots + \epsilon_q(x_{2q-1}^2 + x_{2q}^2) + \lambda_{2q+1}x_{2q+1}^2 + \dots + \lambda_n x_n^2$$

where ϵ_j 's and λ_j 's are equal to ± 1 and the total number of occurrence of -1 is equal to the Morse index of p .

The vector field v and its dual with respect to g take the following form in this coordinate system

$$(4.3) \quad v = (-m_1x_2, m_1x_1, \dots, -m_qx_{2q}, m_qx_{2q-1}, 0, \dots, 0),$$

$$(4.4) \quad v^* = -m_1x_2dx_1 + m_1x_1dx_2 - \dots - m_qx_{2q}dx_{2q-1} + m_qx_{2q-1}dx_{2q}$$

Concerning a non-trivial orbit which is a critical level set of f , There is a coordinates system $(\psi, x_1, x_2, \dots, x_{2q-1}, x_{2q}, x_{2q+1}, \dots, x_n)$ around it such that $o = (\psi, 0, 0, \dots, 0)$; where ψ is considered modulo 2π ; and the following conditions are satisfied

- The metric has the form $g = s^2d\psi^2 + dx_1^2 + dx_2^2 + \dots + dx_n^2$.
- The action has the following representation

$$(4.5) \quad e^{i\theta}(\psi, x_1 + ix_2, \dots, x_n) = (m\theta + \psi, e^{im_1\theta}(x_1 + ix_1), \dots, e^{im_q\theta}(x_{2q-1} + ix_{2q}), x_{2q+1}, \dots, x_n)$$

- The function f is given as follows

$$(4.6) \quad f(\psi, x_1, \dots, x_n) = \epsilon_1(x_1^2 + x_2^2) + \dots + \epsilon_q(x_{2q-1}^2 + x_{2q}^2) + \lambda_{2q+1}x_{2q+1}^2 + \dots + \lambda_n x_n^2$$

Here again the restriction of f to \mathbb{R}^n factor is a Morse function with critical point 0 and takes the standard form (4.2). With respect to these coordinates the vector field v and its dual are as follows

$$(4.7) \quad v = (m, -m_1x_2, m_1x_1, \dots, -m_qx_{2q}, m_qx_{2q-1}, 0, \dots, 0),$$

$$(4.8) \quad v^* = md\psi - m_1x_2dx_1 + m_1x_1dx_2 - \dots - m_qx_{2q}dx_{2q-1} + m_qx_{2q-1}dx_{2q}$$

In each one of above cases, the Clifford hessian of (3.7) takes the following form (see [11, page])

$$(4.9) \quad H = \sum_{i=1}^n \lambda_i Z_i ; \quad Z_i = [dx_i \wedge, dx_i \lrcorner]$$

where $\lambda_i = \pm 1$ is the coefficient of x_i in (4.2) or (4.6).

Remark 1. Z_i at each point is a diagonalizable linear map on exterior algebra generated by dx_i 's. An elements of the form $dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_j}$ is an eigenvector with eigenvalue 1 if $i = i_\ell$ for one ℓ , otherwise the eigenvalues equals -1 .

For next uses we need to review some spectral properties of *harmonic oscillator operator*

$$(4.10) \quad -\frac{d^2}{dx^2} + a^2x^2 ; \quad a > 0$$

In following remark we summarize those properties of this operator that we will need.

Remark 2. The harmonic oscillator operator (4.10) is an unbounded self adjoint operator on $L^2(\mathbb{R})$ and provides a spectral resolution for this Hilbert space. The eigenvalues of this operator are $a(1+2p)$ where $p = 0, 1, 2, \dots$. The eigenvector corresponding to the minimal eigenvalue a is the following function

$$(4.11) \quad W_a(x) := \sqrt[4]{a^2\pi} \exp(-ax^2/2)$$

Given a compactly supported smooth function β on \mathbb{R} such that $\beta(0) = 1$ then

$$(4.12) \quad \lim_{a \rightarrow \infty} \langle \beta(x) W_a(x) ; W_a(x) \rangle = \beta(0) = 1$$

Now we turn our attention toward a critical point p . We assume the coordinates introduced just before 4.1 for a ϵ -neighbourhood $N_\epsilon(p)$ of p . Using (3.6), (4.2) and (4.9) the deformed Laplacian $\Delta_{eq}^k(M)$ coincides with the following operator in a small neighbourhood of p .

$$(4.13) \quad L_s^k : \Omega_{eq}^k(\mathbb{R}^n) \rightarrow \Omega_{eq}^k(\mathbb{R}^n)$$

$$L_s^k = \mathbb{H}_s + A_1 + B_1 + A_2 + B_2 + \dots + A_q + B_q$$

where

$$\mathbb{H}_s = \sum_{\ell=2q+1}^n \frac{\partial^2}{\partial z_\ell^2} + s^2 z_\ell^2 + \lambda_\ell s Z_\ell$$

and for $i = 1, \dots, q$ we have

$$(4.14) \quad A_i = -\frac{\partial^2}{\partial x_{2i-1}^2} - \frac{\partial^2}{\partial x_{2i}^2} + s^2 x_{2i-1}^2 + s^2 x_{2i}^2 + (v_i^* \wedge i_{v_i} + \epsilon_{0,i} i_{v_i} v_i^* \wedge)$$

$$(4.15) \quad B_i = \epsilon s (Z_{2i-1} + Z_{2i}) + 2t \otimes dv_i^* \lrcorner + \epsilon_{0,i} 2t^{i-1} \otimes dv_i^* \wedge$$

Here $\epsilon_i = \pm 1$ is introduced by (4.2) and $v_i = m_i(-x_{2i}, x_{2i-1})$ and $v_i^* = 2m_i dx_{2i-1} \wedge dx_{2i}$. Because \mathbb{H}_s is t -linear, we consider L_s^k as an operator on $\Omega^*(\mathbb{R}^{n-2q}) \otimes \Omega_{eq}^*(\mathbb{R}^{2q})$. Then \mathbb{H}_s acts on the first factor and commute with A_i 's and B_i 's that act on the second factor. If we consider \mathbb{H}_s as an operator on $\Omega^m(\mathbb{R}^{n-2q})$, then its spectrum; by remarks 1 and 2; consists of the following numbers

$$2s \sum_{\ell=2q+1}^n ((1+2p_\ell) + \lambda_\ell q_\ell)$$

Here $p_\ell = 0, 1, 2, \dots$ and $q_\ell = \pm 1$. It is clear that the non-zero element of the spectrum are bounded from below by s , while 0 belongs to spectrum only if for all values of ℓ we have $\lambda_\ell q_\ell = -1$. This is possible only if the Morse index of the critical point $p = 0$ in \mathbb{R}^{n-2q} -direction is equal to m . In this case the kernel is generated by the following element

$$(4.16) \quad W_s(x_{2q+1})W_s(x_{2q+2}) \dots W_s(x_n) dx_{j_1} \wedge dx_{j_2} \wedge \dots dx_{j_m}$$

Here W_s 's are defined by (4.11) and j 's are characterized by the property that $\lambda_j = -1$ in (4.2).

Because for $i = 1, 2, \dots, q$ the operators $A_i + B_i$'s commute with each other, to determine the spectrum of $\sum_i A_i + B_i$ on $\Omega_{eq}^*(\mathbb{R}^{2q})$ we need just to determine the spectrum of $A_i + B_i$ on $\Omega_{eq}^*(\mathbb{R}^2)$ (here $\mathbb{R}^2 = (x_{2i-1}, x_{2i})$). The operator B_i vanishes on $\mathbb{C}[t] \otimes \Omega_G^1(\mathbb{R}^2)$, while A_i is the sum of two uncoupled harmonic oscillators and a non-negative operator. Therefore; by remark 2; its spectrum is bounded from below by $2s$. Consequently the only invariant subspace that might have a non-zero contribution in $\text{tr } \phi(L_s)$ when s goes to infinity is

$$\Omega^*(\mathbb{R}^{n-2q}) \otimes_{\mathbb{C}} \Omega_{eq}^{2*}(\mathbb{R}^2) \otimes \Omega_{eq}^{2*}(\mathbb{R}^2) \otimes \dots \otimes \Omega_{eq}^{2*}(\mathbb{R}^2)$$

Here the i -th \mathbb{R}^2 in above is the two dimensional Euclidean space with coordinates x_{2i-1} and x_{2i} . To determine the spectrum of $A_i + B_i$ on $\Omega_{eq}^{2*}(\mathbb{R}^2)$ we write this space as the direct sum of following invariant orthogonal subspaces

$$(4.17) \quad \Omega_{eq}^{2*}(\mathbb{R}^2) = \Omega_G^0(\mathbb{R}^2) \oplus (\mathbb{C}[t] \otimes \Omega_{eq}^2(\mathbb{R}^2))$$

On the first summand $\Omega_G^0(\mathbb{R}^2)$ we have $Z_i = -\text{Id}$, so

$$(4.18) \quad A_i + B_i = -\frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial y_i^2} + s^2(x_i^2 + y_i^2) - 2\epsilon_i s$$

The spectrum of this operator on $\Omega^0(\mathbb{R}^2)$ consists of numbers $2s(1 + 2p_\ell - \epsilon_i)$ for $p_\ell = 0, 1, 2, \dots$ (see remark 2) whose non-zero elements go to infinity when s does so. Therefore this is also true for the non-zero elements of its spectrum when restricted to $\Omega_G^0(\mathbb{R}^2)$. On the other hand, again by remarks 1 and 2, just when $\epsilon_i = +1$ the spectrum of $A_i + B_i$ contains 0; by choosing $p_\ell = 0$; and the corresponding eigen-function is the following function which is clearly S^1 -invariant

$$(4.19) \quad W_{i,s} := W_s(x_{2i-1})W_s(x_{2i})$$

Here W is defined by (4.11) and we will use this result soon.

By (4.14) and (4.15) the operator $A_i + B_i$ on the second summand in (4.17) is $\mathbb{C}[t]$ -linear. Therefore we need just study its spectrum on $\Omega_{eq}^2(\mathbb{R}^2) = \Omega_G^0(\mathbb{R}^2) \otimes H_i$, where H_i is the linear spaces generated by t , and η_i . Here $\eta_i = r_i^{-1} dr_i \wedge d\psi_i$ and (r_i, ψ_i) are polar coordinates associated to (x_i, y_i) . The operator $A_i + B_i$ on this space has the following expression

$$(4.20) \quad A_i + B_i = -\frac{\partial^2}{\partial x_{2i-1}^2} - \frac{\partial^2}{\partial x_{2i}^2} + (s^2 + m_i^2)(x_{2i-1}^2 + x_{2i}^2) + \begin{bmatrix} -2\epsilon s & 2m_i \\ 2m_i & 2\epsilon s \end{bmatrix}$$

The above matrix acts on H_i and its action commutes with the harmonic oscillator operator defined by other terms. The eigenvalues of the matrix are $\sqrt{s^2 + m_i^2}$ and $-\sqrt{s^2 + m_i^2}$. These correspond, respectively, to the normalized eigenvectors $u_{i,s} := a_i^{-1/2}(m_i t + (\epsilon_i s + \sqrt{s^2 + m_i^2})\eta_i)$ and $w_{i,s} := b_i^{-1/2}(m_i t + (\epsilon_i s - \sqrt{s^2 + m_i^2})\eta_i)$ where $a_i = m_i^2 + (\epsilon_i s + \sqrt{s^2 + m_i^2})^2$ and $b_i = m_i^2 + (\epsilon_i s - \sqrt{s^2 + m_i^2})^2$. With respect to the base $\{u_{i,s}, w_{i,s}\}$ the expression (4.20) takes the following form

$$(4.21) \quad A_i + B_i = -\frac{\partial^2}{\partial x_{2i-1}^2} - \frac{\partial^2}{\partial x_{2i}^2} + (s^2 + m_i^2)(x_{2i-1}^2 + x_{2i}^2) + 2 \begin{bmatrix} \sqrt{s^2 + m_i^2} & 0 \\ 0 & -\sqrt{s^2 + m_i^2} \end{bmatrix}$$

The smallest eigenvalue of the harmonic oscillator (whose corresponding eigenfunction is S^1 -invariant as we have mentioned in above) is $2\sqrt{s^2 + m_i^2}$. Therefore the non-zero eigenvalues of $A_i + B_i$ go to infinity when s goes toward infinity. However 0 always belongs to the spectrum of this operator and its corresponding normalized eigenvector is (see (4.11))

$$\bar{W}_{i,s}(x_{2i-1}, x_{2i}) = W_{s'}(x_{2i-1})W_{s'}(x_{2i}) \otimes w_{i,s}; \quad s'^2 = s^2 + m_i^2$$

Therefore the kernel of the restriction of $A_i + B_i$ to the second summand in (4.17) is generated by $\bar{W}_{i,s}(x_{2i-1}, x_{2i})$.

Now in (4.2) let the number of occurrence of $\epsilon_i = -1$ be k_1 while the number of occurrence of $\lambda_\ell = -1$ be k_2 (so the Morse index of p is $2k_1 + k_2$). Without loss of generality we may assume that ϵ_i 's and λ_ℓ 's are sorted increasingly and we consider the operator L_s^k on $\Omega_{eq}^k(\mathbb{R}^n)$ where $k = 2k_1 + k_2 + \ell$. Following what we have discussed above, the following vectors generated the kernel of L_s^k

$$\bar{W}_{1,s} \dots \bar{W}_{k_1,s} \tilde{W}_{k_1+1,s} \dots \tilde{W}_{q,s} W_s(x_{2q+1}) \dots W_s(x_n) \otimes dx_{2q+1} \wedge \dots \wedge dx_{2q+k_2}$$

Here $\tilde{W}_{k_1+i,s}$ is either equal to $\bar{W}_{k_1+i,s}$ or equal to $W_{k_1+i,s}$. When s goes to infinity the following relations hold

$$\begin{cases} \lim \bar{W}_{i,s} - W_s(x_{2i-1})W_s(x_{2i})\eta_i = 0 & \text{if } \epsilon_i = -1 \\ \lim \bar{W}_{i,s} - W_s(x_{2i-1})W_s(x_{2i}) \otimes t = 0 & \text{if } \epsilon_i = +1 \\ W_{i,s}(x_{2i-1}, x_{2i}) = W_s(x_{2i-1})W_s(x_{2i}) & \text{if } \epsilon_i = +1 \end{cases}$$

Therefore the difference between above vector and the following one goes to zero when s go to infinity

$$(4.22) \quad \Phi := t^m \otimes W_s(x_1)W_s(x_2) \dots W_s(x_n) \otimes \eta_1 \wedge \eta_2 \dots \wedge \eta_{k_1} \wedge dx_{2q+1} \wedge \dots \wedge dx_{2q+k_2}$$

Here $m = q - k_1$ is the number of position where $\epsilon_i = +1$ and we have chosen the kernel of $A_i + B_i$ be an element in $\Omega_{eq}^2(\mathbb{R}^2)$. The exterior-algebraic-grade of this vector is $2k_1 + k_2$ which is equal to the Morse index of the critical point p . Consequently the critical point p contribute in $\text{tr } \phi(L_s^k)$ if and only if its Morse index equals $k, k-2, k-4, \dots$. This proves the following relation

$$(4.23) \quad \lim_{s \rightarrow \infty} \sum_p \text{tr } \phi(L_s^k) = c_k + c_{k-2} + c_{k-4} + \dots$$

We recall from discussion right before corollary 3.4 the compactly supported smooth functions ϕ_p on \mathbb{R}^n with $\phi_p(0) = 1$. As we have discussed in above, all non-zero eigenvalues of L_s^k go to infinity with s and the normalized generator of $\ker L_s^k$ tends toward orthonormal set of vectors given by (4.22). This fact and relation (4.12) imply together the following relation

$$(4.24) \quad \lim_{s \rightarrow \infty} \text{tr } \phi(L_s^k) = \lim_{s \rightarrow \infty} \text{tr } \phi_p \phi(L_s^k)$$

By combining this relation with (4.23) we get the following lemma

Lemma 4.2. *Let ϕ_p be a compactly supported smooth function on \mathbb{R}^n with $\phi_p(0) = 1$. The following relation holds*

$$\lim_{s \rightarrow \infty} \sum_p \text{tr } \phi_p \phi(L_s^k) = c_k + c_{k-2} + c_{k-4} + \dots$$

where p runs over the critical points of the Morse function f and c_i denotes the number of critical points with index i .

Now we compute the contribution of the critical orbits in the trace of $\phi(\Delta_{eq,s}^k)$. In coordinates $(\psi, x_1, x_2, \dots, x_n)$ introduced in above, the deformed equivariant Laplacian takes the following form when acting on $\Omega_{eq}^k(S^1 \times \mathbb{R}^{n-1})$

$$(4.25) \quad \mathcal{L}_s^k = L_s + T$$

where L_s is the deformed equivariant Laplacian on $\Omega_{eq}^*(\mathbb{R}^n)$ given by (4.13) while T is an operator acting on $\Omega_{eq}^*(S^1)$ given by the following expression

$$T = d\psi \wedge i_{\partial/\partial\psi} + \epsilon_{0,i} i_{\partial/\partial\psi} d\psi \wedge.$$

Here we have used the fact that invariant differential forms on S^1 are generated (over \mathbb{C}) by 1 and $d\psi$. We have $\Omega_{eq}^*(S^1 \otimes \mathbb{R}^n) = \Omega_{eq}^*(S^1) \otimes \Omega_{eq}^*(\mathbb{R}^n)$. Here the tensor product is taken over $\mathbb{C}[t]$. Put $\Omega_{eq}^*(S^1) = \mathbb{C}\langle 1 \rangle \oplus \Omega_{eq}^{\geq 1}(S^1)$. Then the first summand belongs to $\ker T$, while the restriction of T to the second summand is given by multiplication by $\|\partial/\partial\psi\|^2 = s^2 m^2$.

As we have seen previously, the non-zero elements of the spectrum of L_s go to infinity with s while the kernel of this operator is asymptotically generated by vectors given by (4.22). Therefore \mathcal{L}_s^k has a non trivial one-dimensional kernel (generated by $1 \otimes \Phi$) only if the power of t in (4.22) is zero. This is possible only if the Morse index of the critical orbit o is k . Therefore if the Morse index of the critical orbit is k then

$$\lim_{s \rightarrow \infty} \text{tr } \phi(\mathcal{L}_s^k) = 1$$

On the other hand, an argument completely similar to the one that led to (4.24) provides the following relation

$$\lim_{s \rightarrow \infty} \text{tr } \phi(\mathcal{L}_s^k) = \lim_{s \rightarrow \infty} \text{tr } \phi_o \phi(\mathcal{L}_s^k)$$

where ϕ_o is the function introduced right before corollary 3.4. Summarizing, we have proved the following lemma

Lemma 4.3. *The following relation holds*

$$\lim_{s \rightarrow \infty} \sum_o \text{tr } \phi_o \phi(\mathcal{L}_s^k) = d_k$$

where o runs over the critical orbits of the Morse function f and d_k denotes the number of critical orbits with index k (of course $d_k = 0$ for $k \geq n$).

Now we have every things to prove the main theorem 2.1.

Proof [of the main theorem 2.1] Actually it follows directly from theorem 3.2 and lemmas 3.5, 4.2 and 4.3. We just need to show that for $k \geq n+1$ the Morse inequalities do not provide new one and reduce to lower order Morse inequalities. We do this for $k = n$, the general case is similar and follows from (3.5). For this purpose note that $d_k = 0$ for $k \geq n$, therefore by 3.5 and 4.2 we have

$$\lim_{s \rightarrow \infty} (\mu_s^{n+1} - \mu_s^n) = (c_{n-1} + c_{n-3} + \dots) - (c_n + c_{n-2} + \dots)$$

The right side of this equality is $(-1)^{n-1}$ times the sum of the indices of the vector field v on its singularities which equals $(-1)^{n-1} \chi(M)$, by the Poincare-Hopf theorem. This and lemma 3.1 show that $\tilde{c}_{n+1} - \tilde{c}_n = \beta_{eq}^{n+1} - \beta_{eq}^n$ and completes the proof. \square

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